

Orlicz-space Hardy and Landau-Kolmogorov inequalities for Gaussian measures

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Abstract

We prove Orlicz-space versions of Hardy and Landau-Kolmogorov inequalities for Gaussian measures on \mathbb{R}^n .

1 Introduction

The classical Hardy inequality on \mathbb{R}^n states that for $u \in W^{1,2}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^2} dx \leq \left(\frac{2}{N-2} \right)^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx, \quad (1.1)$$

which can be written as

$$\left\| u(\cdot) \frac{1}{|\cdot|} \right\|_2 \leq \frac{2}{N-2} \|\nabla u\|_2.$$

It is a natural question to ask for its generalisations: the ‘measure’ $\frac{1}{|x|^2} dx$ on the left hand side of (1.1) can be replaced by $d\mu$, second norm by p -th or q -th, the measure dx on the right hand side by $d\nu$.

For $n = 1$ and functions u vanishing on the boundary, the Hardy inequality (for general measures on $[a, \infty)$) in L^p norms has been thoroughly studied and there is a complete description of measures that allow for such an inequality. We have the following characterization, which can be found in ([9], Section 1.3.1, Th. 1):

Theorem 1.1 ([9]). *Suppose that μ, ν are nonnegative measures on (a, ∞) , let ν^* be the absolutely continuous part of ν . Then the inequality*

$$\left(\int_a^\infty \left| \int_a^x f(t) dt \right|^q d\mu(x) \right)^{\frac{1}{q}} \leq C \left(\int_a^\infty |f(x)|^p d\nu(x) \right)^{\frac{1}{p}}, \quad (1.2)$$

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where $1 \leq p \leq q \leq \infty$, holds for all Borel measurable functions f if and only if

$$B = \sup_{r>a} [\mu([r, \infty))]^{\frac{1}{q}} \left(\int_a^r \left(\frac{d\nu^*}{dx} \right)^{\frac{-1}{p-1}} \right)^{\frac{p-1}{p}} < \infty. \quad (1.3)$$

We are concerned with generalisations of (1.1), when the Lebesgue measure is replaced by the standard Gaussian measure on \mathbb{R}^n , $\gamma_n(dx) = (2\pi)^{-n/2} \exp(-|x|^2/2) dx$, the ‘inner’ weight $w(x) = |x|^{-1}$ is replaced by $w(x) = |x|$, and the L^p -norms are replaced by Orlicz norms or Orlicz modular expressions. Inequalities for the Gaussian measure on \mathbb{R}^n can be reduced to inequalities on $[0, \infty)$, with respect to the measure $d\mu_n(r) = r^{n-1} e^{-r^2/2} dr$. Applying Theorem 1.1 with $d\mu(r) = r^q d\mu_n(r)$ and $d\nu(r) = d\mu_n(r)$, we see that in that case inequality (1.2) with $p = q$ (for Hardy transforms) can hold only if $p > n$.

Another reason why inequalities (1.2) need to be extended is that we need inequalities for measures μ_n holding true not only for Hardy transforms, but also for functions not necessarily vanishing at zero.

More precisely, in this paper we aim at obtaining inequalities of the form:

$$\int_0^\infty M(|rv(r)|) d\mu_n(r) \leq C_1 \int_0^\infty M(|v(r)|) d\mu_n(r) + C_2 \int_0^\infty M(|\nabla v(r)|) d\mu_n(r), \quad (1.4)$$

which then give rise to

$$\int_{\mathbb{R}^n} M(|xv(x)|) d\gamma_n(x) \leq C_1 \int_{\mathbb{R}^n} M(|v(x)|) d\gamma_n(x) + C_2 \int_{\mathbb{R}^n} M(|\nabla v(x)|) d\gamma_n(x), \quad (1.5)$$

with C_1, C_2 independent of v from a sufficiently large class of functions, but depending on the dimension n . This dependence cannot be suppressed, see the discussion at the end of Section 3. We still call inequality (1.5) the Hardy inequality for the Gaussian measure. The Hardy inequality for Gaussian measures are of separate interest, both in the probability theory and the PDE theory. For other inequalities for the Gaussian measure (Poincaré, log-Sobolev) the reader can consult e.g. [8], while in [2, 3, 4] one can find the results concerning the importance of Gaussian measures in the PDE theory.

We obtain (1.4) and (1.5) for general N -functions M satisfying the Δ_2 -condition (doubling), see Proposition 3.2. With some additional condition (close to the property that $M(r)/r^2$ is non-decreasing) we were able to provide a more detailed analysis of the resulting constants.

Inequalities (1.5) are an example of the so-called U -bounds (see [5]), i.e. inequalities of the form

$$\int |v|^q U d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu$$

analyzed in the context of general metric spaces and metric gradients. Examples furnished in that paper indicate that the most interesting U -bounds for a measure $d\mu(x) = e^{-\varphi(x)}$ are those with $U(x) = |\nabla \varphi|$. Such inequalities can be related e.g. to Poincaré,

log-Sobolev and other inequalities for μ . Since for the Gaussian measure one has $\varphi(x) = |x|^2/2$, and $|\nabla\varphi(x)| = |x|$, the weight $w(x) = |x|$ in (1.5) is the most desirable one. In a somewhat different context, such inequalities were also investigated in [4].

As an application we show, using a general theorem from [7], that inequality (1.5) implies the Orlicz version of the Landau-Kolmogorov inequality for the Gaussian measure:

$$\|\nabla u\|_{L^M(\mathbb{R}^n, \gamma_n)} \leq C_1 \|u\|_{L^M(\mathbb{R}^n, \gamma_n)} + C_2 \|\nabla^{(2)} u\|_{L^M(\mathbb{R}^n, \gamma_n)}, \quad (1.6)$$

together with its modular counterpart.

In [6], one proves additive Gagliardo-Nirenberg inequalities in weighted Orlicz spaces. In particular, the following inequality for Gaussian measures was obtained:

$$\|\nabla u\|_{L^M(\mathbb{R}^n, \gamma_n)} \leq C_1 \|u\|_{L^{\Phi_1}(\mathbb{R}^n, \gamma_n)} + C_2 \|\nabla^{(2)} u\|_{L^{\Phi_2}(\mathbb{R}^n, \gamma_n)},$$

where M was an N -function satisfying the Δ_2 -condition and increasing faster than r^2 , and Φ_1, Φ_2 were other N -functions. The functions M, Φ_1, Φ_2 were tied by certain consistency conditions, which in particular *excluded* the case $M = \Phi_1 = \Phi_2$, i.e. the results of [6] did not yield the Landau-Kolmogorov inequality (1.6) in Orlicz norms. This is rectified in present paper, see Corollary 4.1.

2 Preliminaries

2.1 Notation

Throughout the paper, the symbol $\nabla^{(2)}u$ denotes the Hessian of a function $u \in C^2(\mathbb{R}^n)$, i.e. the matrix $[\frac{\partial^2 u}{\partial x_i \partial x_j}]_{i,j=1}^n$. For a square $n \times n$ matrix A , by $|A|$ we denote its Hilbert-Schmidt norm:

$$|A| = |A|_{HS} = \left(\sum_{i,j=1}^n a_{ij}^2 \right)^{\frac{1}{2}};$$

$C_0^\infty(\mathbb{R}^n)$ stands for smooth compactly supported functions on \mathbb{R}^n .

2.2 N -functions

A function $M : [0, \infty) \rightarrow [0, \infty)$ is called an N -function if it is convex, $M(0) = 0$, $\lim_{r \rightarrow 0^+} M(r)/r = 0$ and $\lim_{r \rightarrow \infty} M(r)/r = \infty$. An N -function M is said to satisfy the Δ_2 -condition if and only if

$$\exists C_M > 1 \forall r > 0 \quad M(2r) \leq C_M M(r). \quad (2.1)$$

If additionally M is differentiable, then the Δ_2 -condition (2.1) is equivalent to the existence of $D_M > 1$ s.t. $M'(r) \leq D_M \frac{M(r)}{r}$, for $r > 0$. Additional conditions on M will be added as needed.

2.3 Weighted Orlicz spaces

Suppose that μ is a positive Radon measure on \mathbb{R}^n and let $M : [0, \infty) \rightarrow [0, \infty)$ be an N -function. The weighted space $L^M(\mu)$ with respect to the measure μ is, by definition, the function space

$$L^M(\mathbb{R}^n, \mu) = L^M(\mu) \stackrel{\text{def}}{=} \left\{ f \text{ measurable: } \int_{\mathbb{R}^n} M\left(\frac{|f(x)|}{K}\right) d\mu(x) \leq 1 \text{ for some } K > 0 \right\},$$

equipped with the Luxemburg norm

$$\|f\|_{L^M(\mu)} = \inf \left\{ K > 0: \int_{\mathbb{R}^n} M\left(\frac{|f(x)|}{K}\right) d\mu(x) \leq 1 \right\}.$$

This norm is complete and turns $L^M(\mu)$ into a Banach space. For $M(r) = r^p$ with $p > 1$, the space $L^M(\mu)$ coincides with the usual $L^p(\mu)$ space.

We recall the following two properties of Young functionals: for every $f \in L^M(\mu)$ we have

$$\|f\|_{L^M(\mu)} \leq \int_{\mathbb{R}^n} M(|f(x)|) d\mu(x) + 1, \quad (2.2)$$

and

$$\int_{\mathbb{R}^n} M\left(\frac{|f(x)|}{\|f\|_{L^M(\mu)}}\right) d\mu(x) \leq 1. \quad (2.3)$$

When M satisfies the Δ_2 -condition, then (2.3) becomes an equality.

For more information on Orlicz spaces the reader may consult e.g. [10].

3 The Hardy inequality for the Gaussian measure

3.1 Inequalities on the real line

We start with inequalities for measures $\mu_n(dr) = r^{n-1}e^{-r^2/2}dr$, $r > 0$, where $n = 1, 2, \dots$. In our approach, we will make the following assumption concerning the function M :

(M) $M : [0, \infty) \rightarrow [0, \infty)$ is nonconstant and there exist $d_M, D_M > 0$ such that M satisfies the inequalities

$$\forall_{r \geq 0, a \geq 1} \quad M(ar) \leq a^{D_M} \cdot M(r) \quad (3.1)$$

and

$$\forall_{r \geq 0, a \in (0,1)} \quad M(ar) \leq a^{d_M} \cdot M(r). \quad (3.2)$$

Then, obviously, $D_M \geq d_M$ and M is an increasing continuous function with $M(0) = 0$, $\lim_{r \rightarrow \infty} M(r) = \infty$, and moreover $r \mapsto r^{-d_M} M(r)$ is non-decreasing.

When we additionally assume that $D_M > 2$ and $d_M \geq 2$, then in particular $\lim_{r \rightarrow 0^+} r^{-2} M(r)$ exists and is finite. Hence by a natural convention we treat $r \mapsto r^{-2} M(r)$ and $r \mapsto r^{-1} M(r)$ as continuous functions on the whole $[0, \infty)$, the latter taking value 0 at 0.

Lemma 3.1. *Assume that M satisfies (M) with $d_M \geq 2$, $D_M > 2$. Then for any $\lambda \geq 1/d_M$ and $r, s \geq 0$ we have*

$$r^{-1} M(r) \cdot s \leq (1 - D_M^{-1})(\lambda D_M)^{-1/(D_M-1)} M(r) + \lambda M(s) \quad (3.3)$$

and

$$r^{-2} M(r) \cdot s^2 \leq (1 - 2D_M^{-1})(\lambda D_M)^{-2/(D_M-2)} M(r) + 2\lambda M(s). \quad (3.4)$$

Proof. Because of the continuity we may and will assume that r and s are strictly positive. Let $\alpha \in \{1, 2\}$ and $\psi_\alpha(u) = u^\alpha - \alpha \lambda u^{D_M}$. Since

$$(1 - \alpha D_M^{-1})(\lambda D_M)^{-\alpha/(D_M-\alpha)} = \sup_{u \in [0,1]} \psi_\alpha(u),$$

by setting $u = s/r$ we rewrite both asserted inequalities in the case $s \leq r$ as

$$\psi_\alpha(u) \leq (1 - \alpha D_M^{-1})(\lambda D_M)^{-\alpha/(D_M-\alpha)} + \alpha \lambda \left(\frac{M(s)}{M(r)} - (s/r)^{D_M} \right),$$

so that they immediately follow from $M(r) = M(rs^{-1} \cdot s) \leq (r/s)^{D_M} M(s)$.

For $s \geq r$ we have $M(r) = M(rs^{-1} \cdot s) \leq (r/s)^{d_M} M(s)$, so by setting $u = s/r$ we reduce our task to proving

$$\forall u \geq 1 \quad u^\alpha \leq (1 - \alpha D_M^{-1})(\lambda D_M)^{-\alpha/(D_M-\alpha)} + \alpha \lambda u^{d_M}.$$

The case $u = 1$ of the above estimate follows by the previous argument, and the proof is finished by observing that $\frac{d}{du} u^\alpha \leq \alpha \lambda \cdot \frac{d}{du} u^{d_M}$ for $u > 1$ because $d_M \geq \alpha$ and $\lambda \geq 1/d_M$. \square

Proposition 3.1. *Assume that a non-constant function $M : [0, \infty) \rightarrow [0, \infty)$ satisfies (M) with $d_M \geq 2$ and $D_M > 2$. Let $n \geq 1$ and $d\mu_n(r) = r^{n-1} e^{-r^2/2} dr$. For a continuous and piecewise C^1 -function $u : [0, \infty) \rightarrow \mathbb{R}$ set*

$$\mathcal{K} = \int_0^\infty M(r|u(r)|) d\mu_n(r), \mathcal{L} = \int_0^\infty M(|u(r)|) d\mu_n(r), \mathcal{M} = \int_0^\infty M(|u'(r)|) d\mu_n(r).$$

Then

$$\mathcal{K} \leq (D_M/d_M)^{D_M/(D_M-2)} \mathcal{L} \leq e^{D_M/2} \cdot \mathcal{L} \quad (3.5)$$

or

$$\mathcal{K} \leq \left(\frac{1}{2} D_M \mathcal{M}^{1/D_M} + \sqrt{\frac{1}{4} D_M^2 \mathcal{M}^{2/D_M} + (D_M + n - 2) \mathcal{L}^{2/D_M}} \right)^{D_M}. \quad (3.6)$$

If additionally $D_M + n \geq e + 2$ (which holds true whenever $n \geq 3$), then

$$\mathcal{K} \leq \left(\frac{1}{2} D_M \mathcal{M}^{1/D_M} + \sqrt{\frac{1}{4} D_M^2 \mathcal{M}^{2/D_M} + (D_M + n - 2) \mathcal{L}^{2/D_M}} \right)^{D_M}.$$

Proof. We have $D_M/d_M \leq D_M/2 \leq e^{\frac{D_M}{2}-1}$, which proves the second inequality of (3.5). Therefore, if $D_M + n - 2 \geq e$ then the right-hand side of (3.6) dominates the right-hand side of (3.5), which proves the last assertion. Hence it suffices to prove that (3.5) or (3.6) holds true. Additionally, let us assume at first that u is compactly supported. By a standard integration by parts argument we obtain

$$\begin{aligned} \mathcal{K} &= - \int_0^\infty M(r|u(r)|) r^{n-2} \frac{d}{dr} (e^{-r^2/2}) dr \\ &\leq D_M \int_0^\infty \frac{M(r|u(r)|)}{r|u(r)|} \cdot |u'(r)| d\mu_n(r) + (D_M + n - 2) \int_0^\infty \frac{M(r|u(r)|)}{(r|u(r)|)^2} \cdot u(r)^2 d\mu_n(r). \end{aligned}$$

We have used the fact that $\frac{M(r|u(r)|)}{r|u(r)|} |u(r)| r^{n-1} e^{-r^2/2} \Big|_0^\infty = 0$ since $M(x)/x = 0$ for $x = 0$, and the fact that (3.1) implies $\limsup_{y \rightarrow x} \frac{M(y) - M(x)}{y - x} \leq D_M \frac{M(x)}{x}$. Now, for any $\lambda, \rho \geq 1/d_M$ we can apply (3.3) to estimate the first summand, and (3.4) to bound the second summand, arriving at

$$\begin{aligned} \mathcal{K} &\leq D_M (1 - D_M^{-1}) (\rho D_M)^{-1/(D_M-1)} \mathcal{K} + D_M \rho \mathcal{M} \\ &\quad + (D_M + n - 2) (1 - 2D_M^{-1}) (\lambda D_M)^{-2/(D_M-2)} \mathcal{K} \\ &\quad + 2(D_M + n - 2) \lambda \mathcal{L}. \end{aligned} \tag{3.7}$$

If $\mathcal{K} \leq (D_M/d_M)^{D_M/(D_M-2)} \mathcal{L}$ then our assertion is trivially satisfied.

If $\mathcal{K} \leq (D_M/d_M)^{D_M/(D_M-1)} \mathcal{M}$ then $\mathcal{K} \leq (D_M/2)^{D_M/(D_M-1)} \mathcal{M} \leq (D_M/2)^{D_M} \mathcal{M}$, and again there is nothing to prove. Hence we may and do assume that

$$\mathcal{K} \geq \max \left((D_M/d_M)^{D_M/(D_M-2)} \mathcal{L}, (D_M/d_M)^{D_M/(D_M-1)} \mathcal{M} \right),$$

so that $\lambda_0 = D_M^{-1} (\mathcal{K}/\mathcal{L})^{(D_M-2)/D_M}$ and $\rho_0 = D_M^{-1} (\mathcal{K}/\mathcal{M})^{(D_M-1)/D_M}$ satisfy $\lambda_0, \rho_0 \geq 1/d_M$. By setting $\lambda = \lambda_0$ and $\rho = \rho_0$ in (3.7) we obtain

$$\mathcal{K} \leq D_M \mathcal{M}^{1/D_M} \mathcal{K}^{(D_M-1)/D_M} + (D_M + n - 2) \mathcal{L}^{2/D_M} \mathcal{K}^{(D_M-2)/D_M},$$

so that

$$\left(\mathcal{K}^{1/D_M} - \frac{1}{2} D_M \mathcal{M}^{1/D_M} \right)^2 \leq \frac{1}{4} D_M^2 \mathcal{M}^{2/D_M} + (D_M + n - 2) \mathcal{L}^{2/D_M},$$

which ends the proof in the case of compactly supported u .

In the general case let $N \geq 1$, $u_N(r) = u(r)$ if $r \in [0, N]$, $u_N(r) = \frac{2N-r}{N} u(r)$ if $r \in [N, 2N]$, and $u_N(r) = 0$ if $r \geq 2N$. Let

$$\mathcal{K}_N = \int_0^\infty M(r|u_N(r)|) d\mu_n(r), \quad \mathcal{L}_N = \int_0^\infty M(|u_N(r)|) d\mu_n(r),$$

$$\mathcal{M}_N = \int_0^\infty M(|u'_N(r)|) d\mu_n(r).$$

Since u_N is compactly supported we have

$$\mathcal{K}_N \leq (D_M/d_M)^{D_M/(D_M-2)} \cdot \mathcal{L}_N$$

or

$$\mathcal{K}_N \leq \left(\frac{1}{2} D_M \mathcal{M}_N^{1/D_M} + \sqrt{\frac{1}{4} D_M^2 \mathcal{M}_N^{2/D_M} + (D_M + n - 2) \mathcal{L}_N^{2/D_M}} \right)^{D_M}.$$

By the Monotone Convergence Theorem we obviously have $\mathcal{K}_N \rightarrow \mathcal{K}$ and $\mathcal{L}_N \rightarrow \mathcal{L}$ as $N \rightarrow \infty$ (note that $|u_N| \nearrow |u|$ and recall that M is non-decreasing). Since there is $u'_N(r) = \frac{2N-r}{N} u'(r) - \frac{1}{N} u(r)$ for all points $r \in (N, 2N)$ at which u is differentiable, we get $|u'_N(r)| \leq |u'(r)| + \frac{1}{N} |u(r)|$ for almost all $r > 0$. Let $A_N = \{r > 0 : u'(r) \text{ exists and } |u(r)| \leq N^{1/2} |u'(r)|\}$ and let $B_N = (0, \infty) \setminus A_N$. If $r \in A_N$ then

$$M(|u'_N(r)|) \leq M\left((1 + N^{-1/2})|u'(r)|\right) \leq \left(1 + N^{-1/2}\right)^{D_M} M(|u'(r)|)$$

whereas for almost all $r \in B_N$ we have

$$M(|u'_N(r)|) \leq M\left((N^{-1} + N^{-1/2})|u(r)|\right) \leq \left(2/\sqrt{N}\right)^{D_M} M(|u(r)|).$$

Hence $\mathcal{M}_N \leq (1 + N^{-1/2})^{D_M} \mathcal{M} + \left(2/\sqrt{N}\right)^{D_M} \mathcal{L} \xrightarrow{N \rightarrow \infty} \mathcal{M}$, and the proof is finished. The argument fails only if $\mathcal{L} = \infty$, but then the main assertion is trivial. \square

We may slightly weaken the assertion of Proposition 3.1 by turning it into a more convenient linear estimate:

$$\mathcal{K} \leq C_1 \mathcal{L} + C_2 \mathcal{M}, \quad (3.8)$$

with positive C_1 and C_2 depending only on n , D_M and d_M . Elementary calculations permit us to obtain e.g.

$$C_1 = 2^{D_M-1} (D_M + n - 2)^{\frac{D_M}{2}}, \quad C_2 = 2^{D_M-1} D_M^{D_M},$$

valid when $D_M + n \geq e + 2$. Also, when we consider $M(r) = r^p$, no restrictions other than that $p > 2$ are required, which follows from a straightforward calculation which uses integration by parts and Hölder's inequality only. See also Corollary 3.1 below.

For $\rho > 1$ let

$$\beta(\rho) = \sup_{w>0} \left\{ \left(\frac{1}{2} w + \sqrt{\frac{1}{4} w^2 + 1} \right)^{D_M} - \rho w^{D_M} \right\}$$

and

$$\gamma(\rho) = \sup_{w>0} \left\{ \left(\frac{1}{2} + \sqrt{\frac{1}{4} + w^2} \right)^{D_M} - \rho w^{D_M} \right\}.$$

Obviously, $\beta(\rho)$ and $\gamma(\rho)$ are finite but they grow to infinity as $\rho \rightarrow 1^+$. By simple considerations involving homogeneity we prove that (3.6) implies

$$\mathcal{K} \leq \beta(\rho) \cdot (D_M + n - 2)^{D_M/2} \mathcal{L} + \rho \cdot D_M^{D_M} \mathcal{M}$$

and

$$\mathcal{K} \leq \rho \cdot (D_M + n - 2)^{D_M/2} \mathcal{L} + \gamma(\rho) \cdot D_M^{D_M} \mathcal{M}.$$

Thus, under assumptions of Proposition 3.1, we may obtain (3.8) with any C_1 greater than $\max((D_M + n - 2)^{D_M/2}, (D_M/d_M)^{D_M/(D_M-2)})$ – note that this quantity does not exceed $(\max(D_M + n - 2, e))^{D_M/2}$. However, this comes at the expense of C_2 getting large. Similarly, under the same assumptions, we may prove (3.8) with any $C_2 > D_M^{D_M}$, at the expense of C_1 getting large.

If M is a power function, $M(r) = r^p$ for some $p > 2$, then $D_M = d_M = p$, and we obtain the following corollary to Proposition 3.1.

Corollary 3.1. *Assume that $n \geq 1$ and $p > 2$. Let $d\mu_n = r^{n-1}e^{-r^2/2} dr$. For an a.e. differentiable function $u : [0, \infty) \rightarrow \mathbb{R}$ let*

$$\mathcal{K} = \int_0^\infty (r|u(r)|)^p d\mu_n(r), \quad \mathcal{L} = \int_0^\infty |u(r)|^p d\mu_n(r), \quad \mathcal{M} = \int_0^\infty |u'(r)|^p d\mu_n(r).$$

Then for every $C_2 > p^p$ there exists some $C_1 = C_1(n, p, C_2) < \infty$, and for every $C_1 > (n + p - 2)^{p/2}$ there exists some $C_2 = C_2(n, p, C_1) < \infty$, such that for every continuous and piecewise C^1 function $u : [0, \infty) \rightarrow \mathbb{R}$ there is

$$\mathcal{K} \leq C_1 \mathcal{L} + C_2 \mathcal{M}.$$

Proof. Preceding considerations. □

Remark 3.1. It is known (see [4]) that for $p = 2$ one has

$$\frac{1}{4} \mathcal{K} \leq \mathcal{L} + \frac{n}{2} \mathcal{M},$$

(i.e. $C_1 = 2n$, $C_2 = 4$) and that the constant $\frac{1}{4}$ cannot be improved. Additionally, if $C_2 = 4$ then (3.8) holds true with $C_1 = 2n$ but it fails for $C_1 < 2n$. In this case ($p = 2$), our method permits to lower C_1 as close to n as we wish, again at the expense of getting C_2 large. Getting $C_1 = n$ is not possible.

The constants C_1 and C_2 in Corollary 3.1 (and thus also the bounds of Proposition 3.1) are quite good. Indeed, assume that (3.8) holds with some constants C_1 and C_2 . Let $\alpha \in [0, 1)$. A straightforward calculation yields that for $u(r) = \exp\left(\frac{\alpha r^2}{2p}\right)$ there is

$$\begin{aligned} \mathcal{K} &= \int_0^\infty (ru_\alpha(r))^p r^{n-1} e^{r^2/2} dr = \int_0^\infty r^{n+p-1} e^{-\frac{(1-\alpha)r^2}{2}} dr = \\ &= (1-\alpha)^{-(n+p)/2} \int_0^\infty \rho^{n+p-1} e^{-\rho^2/2} d\rho = (1-\alpha)^{-(n+p)/2} 2^{(n+p-2)/2} \Gamma\left(\frac{n+p}{2}\right), \end{aligned}$$

$$\begin{aligned}\mathcal{L} &= \int_0^\infty (u_\alpha(r))^p r^{n-1} e^{-r^2/2} dr = (1-\alpha)^{-n/2} 2^{(n-2)/2} \Gamma\left(\frac{n}{2}\right), \\ \mathcal{M} &= \int_0^\infty (u'_\alpha(r))^p r^{n-1} e^{-r^2/2} dr = (\alpha/p)^p (1-\alpha)^{-(n+p)/2} 2^{(n+p-2)/2} \Gamma\left(\frac{n+p}{2}\right).\end{aligned}$$

so that

$$(1-\alpha)^{-p/2} 2^{p/2} \Gamma\left(\frac{n+p}{2}\right) \leq C_1 \Gamma\left(\frac{n}{2}\right) + C_2 (\alpha/p)^p (1-\alpha)^{-p/2} 2^{p/2} \Gamma\left(\frac{n+p}{2}\right), \quad (3.9)$$

Were $C_2 \leq p^p$, (3.9) would imply that $C_1 \geq \frac{2^{p/2} \Gamma((n+p)/2)}{\Gamma(n/2)} \frac{1-\alpha^p}{(1-\alpha)^{p/2}} \rightarrow \infty$ as $\alpha \rightarrow 1^-$. The obtained contradiction proves that in general (3.8) cannot hold with $C_2 \leq p^p$.

Moreover, by taking $\alpha = 0$ in (3.9) we immediately see that in general (3.8) cannot hold with $C_1 < 2^{p/2} \Gamma((n+p)/2) / \Gamma(n/2)$. Note that by Stirling's formula we have

$$\lim_{n \rightarrow \infty} \frac{2^{p/2} \Gamma((n+p)/2)}{(n+p-2)^{p/2} \Gamma(n/2)} = 1,$$

so the assumption that $C_1 > (n+p-2)^{p/2}$ in Corollary 3.1 is also (asymptotically) quite tight.

Proposition 3.1 provides reasonable bounds but its assumptions are a bit restrictive in that they require the function $r \mapsto r^{-2} M(r)$ to be non-decreasing. However, we may also prove (3.8)-type inequality if we replace (3.2) by convexity. This time we do not push for the best possible constants.

Proposition 3.2. *Let $M : [0, \infty) \rightarrow [0, \infty)$ be an increasing and convex function with $M(0) = 0$. Assume that for some $D_M > 0$, $M(\alpha x) \leq \alpha^{D_M} M(x)$ for any $\alpha \geq 1$, $x \geq 0$ (doubling). For $n \geq 1$ let $d\mu_n(r) = r^{n-1} e^{-r^2/2} dr$. Then for any continuous, piecewise C^1 function $u : [0, \infty) \rightarrow \mathbb{R}$ we have*

$$\int_0^\infty M(r|u(r)|) d\mu_n(r) \leq C_1 \cdot \int_0^\infty M(|u(r)|) d\mu_n(r) + C_2 \cdot \int_0^\infty M(|u'(r)|) d\mu_n(r), \quad (3.10)$$

with C_1 and C_2 depending only on n and D_M .

Since M is convex and increasing there must be $D_M \geq 1$. Observe that when M is an N -function satisfying the Δ_2 -condition, then the assumptions of Proposition 3.2 are satisfied.

We need a simple lemma.

Lemma 3.2. *For every $\varepsilon \in (0, 1]$ and every $a, b \geq 0$ we have*

$$M(a)b \leq \varepsilon M(a) + \varepsilon^{-D_M} M(ab).$$

Proof. If $b \geq 1$ then

$$M(a) = M(b^{-1} \cdot ab + (1 - b^{-1}) \cdot 0) \leq b^{-1} M(ab) + (1 - b^{-1}) M(0) = b^{-1} M(ab)$$

and the inequality obviously holds. For $b \in [0, \varepsilon]$ the inequality is trivial. For $b \in (\varepsilon, 1)$ we have

$$M(a)b \leq M(a) = M(b^{-1} \cdot ab) \leq (1/b)^{D_M} M(ab) \leq \varepsilon^{-D_M} M(ab).$$

□

Proof of Proposition 3.2. Again, we first assume additionally that u is compactly supported. Let

$$\mathcal{K} = \int_0^\infty M(r|u(r)|) d\mu_n(r), \mathcal{L} = \int_0^\infty M(|u(r)|) d\mu_n(r), \mathcal{M} = \int_0^\infty M(|u'(r)|) d\mu_n(r).$$

For any $\kappa \geq 1$ we have $\mathcal{L} \geq e^{-(\kappa+1)^2/2} \cdot \int_\kappa^{\kappa+1} M(|u(r)|) dr$, so that there exists $\tilde{r} \in [\kappa, \kappa+1]$ such that $M(|u(\tilde{r})|) \leq e^{2\kappa^2} \mathcal{L}$. We also have

$$\begin{aligned} M\left(\int_\kappa^{\kappa+1} |u'(r)| dr\right) &\leq \int_\kappa^{\kappa+1} M(|u'(r)|) dr \leq \\ e^{(\kappa+1)^2/2} \int_\kappa^{\kappa+1} M(|u'(r)|) r^{n-1} e^{-r^2/2} dr &\leq e^{2\kappa^2} \mathcal{M}. \end{aligned}$$

Hence

$$\begin{aligned} M(|u(\kappa)|) &\leq M\left(|u(\tilde{r})| + \int_\kappa^{\kappa+1} |u'(r)| dr\right) \leq M\left(2 \max\left(|u(\tilde{r})|, \int_\kappa^{\kappa+1} |u'(r)| dr\right)\right) \leq \\ 2^{D_M} \max\left(M(|u(\tilde{r})|), M\left(\int_\kappa^{\kappa+1} |u'(r)| dr\right)\right) &\leq 2^{D_M} e^{2\kappa^2} \max(\mathcal{L}, \mathcal{M}) \leq 2^{D_M} e^{2\kappa^2} (\mathcal{L} + \mathcal{M}). \end{aligned}$$

We have

$$\begin{aligned} \mathcal{K} &= \int_0^\kappa M(r|u(r)|) r^{n-1} e^{-r^2/2} dr + \int_\kappa^\infty M(r|u(r)|) r^{n-1} e^{-r^2/2} dr \leq \\ \kappa^{D_M} \mathcal{L} - \int_\kappa^\infty M(r|u(r)|) r^{n-2} \frac{d}{dr}(e^{-r^2/2}) dr &\leq \\ \kappa^{D_M} \mathcal{L} + M(\kappa|u(\kappa)|) \kappa^{n-2} e^{-\kappa^2/2} + (n-2) \int_\kappa^\infty M(r|u(r)|) r^{n-3} e^{-r^2/2} dr &+ \\ D_M \int_\kappa^\infty \frac{M(r|u(r)|)}{r|u(r)|} (|u(r)| + r|u'(r)|) r^{n-2} e^{-r^2/2} dr &\leq \\ \kappa^{D_M} \mathcal{L} + 2^{D_M} e^{2\kappa^2} \kappa^{D_M+n-2} (\mathcal{L} + \mathcal{M}) + \kappa^{-2} (D_M + n - 2) \mathcal{K} &+ \\ D_M \int_{(\kappa, \infty) \cap \{u \neq 0\}} M(r|u(r)|) \left| \frac{u'(r)}{ru(r)} \right| r^{n-1} e^{-r^2/2} dr &\stackrel{\text{Lem. 3.2}}{\leq} \\ \kappa^{D_M} \mathcal{L} + 2^{D_M} e^{2\kappa^2} \kappa^{D_M+n-2} (\mathcal{L} + \mathcal{M}) + \kappa^{-2} (D_M + n) \mathcal{K} &+ \varepsilon D_M \mathcal{K} + \varepsilon^{-D_M} D_M \mathcal{M}. \end{aligned}$$

By setting $\varepsilon = (4D_M)^{-1}$ and $\kappa = 2(D_M + n)^{1/2}$, upon obvious cancellations we obtain the asserted estimate. Finally, we may remove the compact support assumption in the same way as in the proof of Proposition 3.1. \square

When M is an N -function satisfying the Δ_2 -condition, then using standard Orlicz-space methods one can obtain the Hardy inequality for norms.

Corollary 3.2. *Suppose that $M : [0, \infty) \rightarrow [0, \infty)$ is an N -function satisfying the Δ_2 -condition. Then the assertion of Proposition 3.2 holds true, and, moreover, there exists a constant $C > 0$ such that for any continuous, piecewise C^1 function $u : [0, \infty) \rightarrow \mathbb{R}$*

$$\|xu(x)\|_{L^M([0, \infty), \mu_n)} \leq C \left(\|u(x)\|_{L^M([0, \infty), \mu_n)} + \|u'(x)\|_{L^M([0, \infty), \mu_n)} \right). \quad (3.11)$$

Proof. We only need to prove (3.11). For short, write $\|u\|_M$ instead of $\|u\|_{L^M([0, \infty), \mu_n)}$. For a given nonconstant function u , consider $\tilde{u} = \frac{u}{\|u\|_M + \|u'\|_M}$, and write (3.10) for function \tilde{u} :

$$\begin{aligned} \int_0^\infty M(|x\tilde{u}(x)|) d\mu_n(x) &\leq C_1 \int_0^\infty M(|\tilde{u}(x)|) d\mu_n + C_2 \int_0^\infty M(|\tilde{u}'(x)|) d\mu_n \\ &\leq C_1 \int_0^\infty M\left(\frac{|u|}{\|u\|_M}\right) d\mu_n + C_2 \int_0^\infty M\left(\frac{|u'|}{\|u'\|_M}\right) d\mu_n \\ &= C_1 + C_2 \end{aligned}$$

(we have used (2.3)). It follows that $\|x\tilde{u}(x)\|_M \leq C_1 + C_2 + 1$, and since $\|\cdot\|_M$ is a norm, (3.11) follows. \square

3.2 The n -dimensional case

Using the one-dimensional inequality as a tool, now we derive the Hardy inequality for the n -dimensional Gaussian measure. We start with the statement under general assumptions on the function M involved, which however does not give a good control on the resulting constants.

Theorem 3.1. *Let $M : [0, \infty) \rightarrow \infty$ be an increasing and convex function with $M(0) = 0$. Assume that $M(ax) \leq a^{D_M} M(x)$ for any $a \geq 1$, $x \geq 0$ (doubling). Let $n \geq 1$, and let $d\gamma_n(x) = e^{-|x|^2/2}$. Then for $u \in C_0^1(\mathbb{R}^n)$ we have:*

$$\int_{\mathbb{R}^n} M(|x||u(x)|) d\gamma_n(x) \leq C_1 \int_{\mathbb{R}^n} M(|u(x)|) d\gamma_n(x) + C_2 \int_{\mathbb{R}^n} M(|\nabla u|) d\gamma_n(x), \quad (3.12)$$

where C_1, C_2 are constants from Proposition 3.2. When M is an N -function, then also

$$\|u(x)x\|_{L^M(\mathbb{R}^n, \gamma_n)} \leq C \left(\|u(x)\|_{L^M(\mathbb{R}^n, \gamma_n)} + \|\nabla u\|_{L^M(\mathbb{R}^n, \gamma_n)} \right). \quad (3.13)$$

Proof. We start with the proof of (3.12). We can write, in spherical coordinates,

$$\int_{\mathbb{R}^n} M(|x| |u(x)|) d\gamma_n(x) = \omega_n \int_{S^{n-1}} \int_0^\infty M(|ru(r, y)|) r^{n-1} e^{-\frac{r^2}{2}} dr d\sigma_{n-1}(y),$$

where σ_{n-1} denotes the normalized surface measure on $S^{n-1} \subset \mathbb{R}^n$, ω_n is the standard $(n-1)$ -dimensional surface measure of S^{n-1} , and $u(r, y) = u(ry)$ (for $n=1$, we do not have to do anything). For y fixed, the function $v(r) = u(r, y)$ is a C^1 -function, and so we can apply Proposition 3.2 for $v(r) = u(r, y)$. We obtain (for given y):

$$\begin{aligned} & \int_0^\infty M(r|v(r)|) r^{n-1} e^{-r^2/2} dr \leq \\ & C_1 \int_0^\infty M(|v(r)|) r^{n-1} e^{-r^2/2} dr + C_2 \int_0^\infty M(|v'(r)|) r^{n-1} e^{-r^2/2} dr, \end{aligned}$$

and the constants C_1, C_2 do not depend on y . Note also that $v'(r) = \frac{\partial u}{\partial r}(r, y)$, and so $|v'(r)| \leq |\nabla u(x)|$. Switching back to Euclidean coordinates we obtain:

$$\int_{\mathbb{R}^n} M(|x| |u(x)|) d\gamma_n(x) \leq C_1 \int_{\mathbb{R}^n} M(|u(x)|) d\gamma_n(x) + C_2 \int_{\mathbb{R}^n} M(|\nabla u(x)|) d\gamma_n(x).$$

(3.12) is proven.

To get (3.13), we use a standard Orlicz-space argument similar to that in the proof of Corollary 3.2: we apply (3.12) to the function $\tilde{u} = \frac{u}{\|u\|_M + \|\nabla u\|_M}$, and then we proceed as before. \square

Under more restrictive assumptions on M , we can use Proposition 3.1 instead of Proposition 3.2, so that the constants are better controlled. For example, when $D_M + n - 2 \geq e$, we get:

Theorem 3.2. *Suppose that M satisfies condition **(M)** with $d_M \geq 2$ and $D_M > \max\{2, e + 2 - n\}$. Let $u \in C_0^1(\mathbb{R}^n)$. Denoting*

$$\begin{aligned} \mathcal{K}^{(n)} &= \int_{\mathbb{R}^n} M(|x| |u(x)|) d\gamma_n(x), \quad \mathcal{L}^{(n)} = \int_{\mathbb{R}^n} M(|u(x)|) d\gamma_n(x) \\ \mathcal{M}^{(n)} &= \int_{\mathbb{R}^n} M(|\nabla u(x)|) d\gamma_n(x), \end{aligned}$$

one gets

$$\mathcal{K}^{(n)} \leq \left(\frac{1}{2} D_M \left(\mathcal{M}^{(n)} \right)^{1/D_M} + \sqrt{\frac{1}{4} D_M^2 \left(\mathcal{M}^{(n)} \right)^{2/D_M} + (D_M + n - 2) \left(\mathcal{L}^{(n)} \right)^{2/D_M}} \right)^{D_M}. \quad (3.14)$$

The proof is identical with that of Theorem 3.1.

4 The Landau-Kolmogorov inequality for the Gaussian measure

The Hardy inequalities from Section 3.2 can be used for deriving Landau-Kolmogorov inequalities for Gaussian measures in \mathbb{R}^n .

To this end, we will use the following theorem (Theorem 3.3 of [7]), applied with $P = Q = M$.

Theorem 4.1 ([7]). *Let $\Omega \subset \mathbb{R}^n$ be an open domain. Suppose that M is a differentiable N -function satisfying the Δ_2 -condition and such that $M(r)/r^2$ is non-decreasing. Let $d\mu(x) = e^{-\varphi(x)}dx$ be a Radon measure on Ω such that $\varphi \in W_{loc}^{1,\infty}(\Omega)$. If for every $u \in C_0^\infty(\Omega)$ the following Hardy-type inequality holds true:*

$$\int_{\Omega} M(|\nabla \varphi| |u|) d\mu \leq K_1 \int_{\Omega} M(|\nabla u|) d\mu + K_2 \int_{\Omega} M(|u|) d\mu, \quad (4.1)$$

then we have:

1) *there exist positive constants C_1, C_2 such that for any $\theta \in (0, 1]$ and any $u \in C_0^\infty(\Omega)$*

$$\int_{\Omega} M(|\nabla u|) d\mu \leq C_1 \int_{\Omega} M(\theta |\nabla^{(2)} u|) d\mu + C_2 \int_{\Omega} M(|u|/\theta) d\mu; \quad (4.2)$$

2) *there exist positive constants \tilde{C}_1, \tilde{C}_2 such that for any $u \in C_0^\infty(\Omega)$*

$$\|\nabla u\|_{L^M(\Omega, \mu)} \leq \tilde{C}_1 \sqrt{\|\nabla^{(2)} u\|_{L^M(\Omega, \mu)} \|u\|_{L^M(\Omega, \mu)}} + \tilde{C}_2 \|u\|_{L^M(\Omega, \mu)}. \quad (4.3)$$

We apply this theorem to $\Omega = \mathbb{R}^n$ and $d\mu(x) = e^{-|x|^2/2}dx$. In this case $|\nabla \varphi(x)| = |x|$, and the validity of (4.1) is assured by Proposition 3.2 (or Proposition 3.1, provided we assume **(M)**). Choosing $\theta = 1$ we obtain the following:

Corollary 4.1. *Suppose M is a differentiable N -function satisfying the Δ_2 -condition and such that $M(r)/r^2$ is non-decreasing. Let $d\gamma_n(x) = e^{-|x|^2/2}dx$. Then there exist positive constants C_1, C_2 such that for any $u \in C_0^\infty(\mathbb{R}^n)$ one has*

$$\int_{\mathbb{R}^n} M(|\nabla u|) d\gamma_n \leq C_1 \int_{\mathbb{R}^n} M(|\nabla^{(2)} u|) d\gamma_n + C_2 \int_{\mathbb{R}^n} M(|u|) d\gamma_n; \quad (4.4)$$

and positive constants \tilde{C}_1, \tilde{C}_2 such that for any $u \in C_0^\infty(\Omega)$

$$\|\nabla u\|_{L^M(\mathbb{R}^n, \gamma_n)} \leq \tilde{C}_1 \sqrt{\|\nabla^{(2)} u\|_{L^M(\mathbb{R}^n, \gamma_n)} \|u\|_{L^M(\mathbb{R}^n, \gamma_n)}} + \tilde{C}_2 \|u\|_{L^M(\mathbb{R}^n, \gamma_n)}. \quad (4.5)$$

By usual density arguments, smoothness conditions on u can be relaxed.

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